TEOREMA DE GREEN

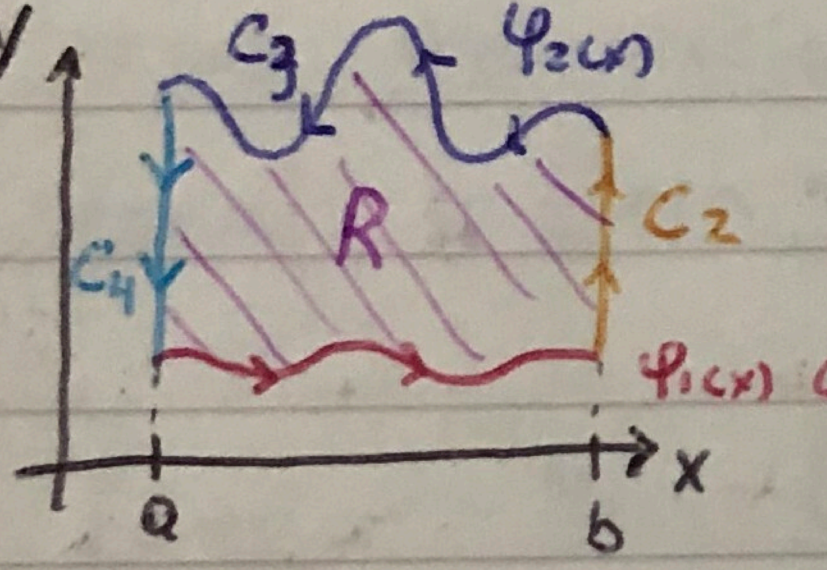
Sea F un campo vectorial / $F=(P_{(x,y)}, Q_{(x,y)})$, $P, Q: A \subseteq \mathbb{R}^2 \to \mathbb{R}$, $F: A \subseteq \mathbb{R}^2 \to \mathbb{R}^2$, $F \in C^1$, C una curva cerrada simple y suave a trozos que encierra un región R de Tipo III / $R \subseteq A \Rightarrow \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy = \int_{C^+} P\,dx + Q\,dy$.

<u>Tomo:</u> $F = F_1 + F_2$, $F_1 = (P, 0)$, $F_2 = (0, Q)$

$$\Rightarrow \int_{C^+} F\,ds = \int_{C^+} F_1\,ds + \int_{C^+} F_2\,ds = \int_{C^+} P\,dx + \int_{C^+} Q\,dy$$

Como R es una región de Tipo III en particular es de Tipo I y entonces puedo escribir R como:

$$R = \{(x,y) \in \mathbb{R}^2 / x \in [a,b], y \in [\varphi_1(x), \varphi_2(x)]\}$$



$$C^+ = C_1^+ + C_2^+ + C_3^+ + C_4^+$$

$$\Rightarrow \int_{C^+} P\,dx = \int_{C_1^+} P\,dx + \int_{C_2^+} P\,dx + \int_{C_3^+} P\,dx + \int_{C_4^+} P\,dx$$

$$\int_{C_1^+} P\,dx = \int_a^b \langle (P_{(x,\varphi_1(x))}, 0), (1, \varphi_1'(x)) \rangle\,dx = \int_a^b P_{(x,\varphi_1(x))}\,dx$$

$$\int_{C_2^+} P\,dx = \int_{\varphi_1(b)}^{\varphi_2(b)} \langle (P_{(b,t)}, 0), (0,1) \rangle\,dx = \int_{\varphi_1(b)}^{\varphi_2(b)} 0\,dx = 0$$

$$\int_{C_3^+} P\,dx = \int_b^a \langle (P_{(x,\varphi_2(x))}, 0), (1, \varphi_2'(x)) \rangle\,dx = -\int_a^b P_{(x,\varphi_2(x))}\,dx$$

$$\int_{C_4^+} P\,dx = \int_{\varphi_2(a)}^{\varphi_1(a)} \langle (P_{(a,t)}, 0), (0,1) \rangle\,dx = -\int_{\varphi_1(a)}^{\varphi_2(a)} 0\,dx = 0$$

$$\Rightarrow \int_{C^+} P\,dx = \int_a^b \left(P_{(x,\varphi_1(x))} - P_{(x,\varphi_2(x))}\right) dx$$

Notemos: $\iint_R \frac{\partial P}{\partial y}\,dx\,dy = \int_a^b\int_{\varphi_1(x)}^{\varphi_2(x)} \frac{\partial P}{\partial y}(x,y)\,dx\,dy = \int_a^b \left(P(x,\varphi_2(x)) - P(x,\varphi_1(x))\right)dx$

Por Teorema Fundamental del Cálculo

$\Rightarrow \int_{C^+} P\,dx = -\iint_R \frac{\partial P}{\partial y}(x,y)\,dx\,dy$

Ahora, tomando R como una región de Tipo II de la forma:

$R = \{(x,y) \in \mathbb{R}^2 \,/\, x \in [\psi_1(y), \psi_2(y)],\ y \in [c,d]\}$

Se demuestra análogamente que: $\int_{C^+} Q\,dy = \iint_R \frac{\partial Q}{\partial x}(x,y)\,dx\,dy$

$\Rightarrow \int_{C^+} F\,ds = \int_{C^+} P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dx\,dy$ Q.E.D.

ÁREA DE UNA SUPERFICIE

Sea C una curva cerrada simple y suave a trozos que encierra una región R de Tipo III, $R \in \mathbb{R}^2$

$\Rightarrow$ $\text{Área}(R) = \frac{1}{2} \int_{C^+} -y\,dx + x\,dy$.

Tomo: $F(x,y) = (-\frac{1}{2}y, \frac{1}{2}x) = (P, Q)$

$$\Rightarrow \int_{C^+} F\,ds = \frac{1}{2}\int_{C^+} -y\,dx + x\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$ Por Teorema de Green.

$$= \iint_R \left(\frac{1}{2} + \frac{1}{2}\right) dx\,dy = \iint_R dx\,dy = \text{Área}(R)$$ Q.E.D.

DIVERGENCIA EN UN PLANO

Sea F un campo vectorial / $F = (P_{(x,y)}, Q_{(x,y)})$, $P, Q: A \subseteq \mathbb{R}^2 \to \mathbb{R}$, $F: A \to \mathbb{R}^2$, $F \in C^1$, C una curva cerrada simple y suave a trozos que encierra una Región de Tipo III R / $R \subseteq A$ con normal unitaria exterior a C^+ $\Rightarrow \int_{C^+} F\,ds = \iint_R \nabla \cdot F\,dx\,dy$.

Tomo: $\alpha_{(t)} = (x_{(t)}, y_{(t)})$ parametrización regular de C^+

$$\Rightarrow \eta = \frac{(y'_{(t)}, -x'_{(t)})}{\|(x'_{(t)}, y'_{(t)})\|}$$ es la normal unitaria exterior a C^+.

$$\Rightarrow \int_{C^+} F\,ds = \int_{C^+} \left\langle F(\alpha_{(t)}), \cancel{\frac{1}{\|\alpha'_{(t)}\|}} (y'_{(t)}, -x'_{(t)}) \right\rangle \cancel{\|\alpha'_{(t)}\|}\, dt$$

$$= \int_{C^+} \left(P_{(x_{(t)}, y_{(t)})}\, y'_{(t)} - Q_{(x_{(t)}, y_{(t)})}\, x'_{(t)}\right) dt = \int_{C^+} \left\langle \left(-Q_{(x_{(t)}, y_{(t)})}, P_{(x_{(t)}, y_{(t)})}\right), (x'_{(t)}, y'_{(t)}) \right\rangle dt$$

$$= \int_{C^+} P\,dy - Q\,dx \overset{\text{Green}}{=} \iint_R \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right) dx\,dy = \iint_R \nabla \cdot F\,dx\,dy$$ Q.E.D.

TEOREMA DE STOKES

Sea $S \subseteq \mathbb{R}^3$; $T: D \subseteq \mathbb{R}^2 \to \mathbb{R}^3$ parametrización regular de S, D unión de regiones Tipo III, $T \in C^2$, ∂D^+ suave a trozos, S orientada por T, $\partial S^+ = T(\partial D^+)$ y $F: S \to \mathbb{R}^3$ un campo vectorial, $F \in C^1 \Rightarrow \iint_S \nabla \times F \, dS = \int_{\partial S^+} F \, ds$.

Tome: $F = (P_{(x,y,z)}, Q_{(x,y,z)}, R_{(x,y,z)})$

qvq.: $\int_{\partial S^+} P\,dx = \iint_S \left(0, \frac{\partial P}{\partial z}, -\frac{\partial P}{\partial y}\right) dS$, $\int_{\partial S^+} Q\,dy = \iint_S \left(-\frac{\partial Q}{\partial z}, 0, \frac{\partial Q}{\partial x}\right) dS$,

$\int_{\partial S^+} R\,dz = \iint_S \left(\frac{\partial R}{\partial y}, -\frac{\partial R}{\partial x}, 0\right) dS$ ya que:

$$\iint_S \nabla \times F \, dS = \iint_S \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}, \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dS$$

$T = (T_1, T_2, T_3)$

$$\iint_S \left(0, \frac{\partial P}{\partial z}, -\frac{\partial P}{\partial y}\right) dS = \iint_D \left\langle \left(0, \frac{\partial P}{\partial z}(T(u,v)), -\frac{\partial P}{\partial y}(T(u,v))\right), \frac{\partial T}{\partial u} \times \frac{\partial T}{\partial v} \right\rangle du\,dv$$

$$* \; \frac{\partial T}{\partial u} \times \frac{\partial T}{\partial v} = \left(\frac{\partial T_2}{\partial u}\frac{\partial T_3}{\partial v} - \frac{\partial T_3}{\partial u}\frac{\partial T_2}{\partial v}, \frac{\partial T_3}{\partial u}\frac{\partial T_1}{\partial v} - \frac{\partial T_1}{\partial u}\frac{\partial T_3}{\partial v}, \frac{\partial T_1}{\partial u}\frac{\partial T_2}{\partial v} - \frac{\partial T_2}{\partial u}\frac{\partial T_1}{\partial v}\right)$$

$$\Rightarrow \iint_S \left(0, \frac{\partial P}{\partial z}, -\frac{\partial P}{\partial y}\right) dS = \iint_D \left(\frac{\partial P}{\partial z}\left(\frac{\partial T_3}{\partial u}\frac{\partial T_1}{\partial v} - \frac{\partial T_1}{\partial u}\frac{\partial T_3}{\partial v}\right) - \frac{\partial P}{\partial y}\left(\frac{\partial T_1}{\partial u}\frac{\partial T_2}{\partial v} - \frac{\partial T_2}{\partial u}\frac{\partial T_1}{\partial v}\right)\right) du\,dv$$

$$\int_{\partial S^+} (P, 0, 0)\, ds = \int_a^b \left\langle \left(P(T(\alpha(t))), 0, 0\right), \left(T_1'(\alpha(t)), T_2'(\alpha(t)), T_3'(\alpha(t))\right) \right\rangle dt$$

$$= \int_a^b \left(P(T(\alpha(t))) \frac{\partial T_1}{\partial u} \alpha_1'(t) + P(T(\alpha(t))) \frac{\partial T_1}{\partial v} \alpha_2'(t)\right) dt$$

$$= \int_a^b \left\langle \left(P \circ T \frac{\partial T_1}{\partial u}, P \circ T \frac{\partial T_1}{\partial v}\right), \left(\alpha_1'(t), \alpha_2'(t)\right) \right\rangle dt$$

$$= \int_{\partial D^+} P\circ T \frac{\partial T_1}{\partial u}\,du + P\circ T \frac{\partial T_1}{\partial v}\,dv \overset{\text{Por Teorema de Green}}{=} \iint_D \left(\frac{\partial (P\circ T \frac{\partial T_1}{\partial v})}{\partial u} - \frac{\partial (P\circ T \frac{\partial T_1}{\partial u})}{\partial v} \right) du\,dv$$

$$= \iint_D \left(\left(\frac{\partial P\circ T}{\partial u}\frac{\partial T_1}{\partial v} + P\circ T \frac{\partial^2 T_1}{\partial u \partial v} \right) - \left(\frac{\partial P\circ T}{\partial v}\frac{\partial T_1}{\partial u} + P\circ T \frac{\partial^2 T_1}{\partial v \partial u} \right) \right) du\,dv$$

$$= \iint_D \left(\frac{\partial P\circ T}{\partial u}\frac{\partial T_1}{\partial v} - \frac{\partial P\circ T}{\partial v}\frac{\partial T_1}{\partial u} + \cancel{P\circ T \left(\frac{\partial^2 T_1}{\partial u \partial v} - \frac{\partial^2 T_1}{\partial v \partial u} \right)} \right) du\,dv \quad \text{Porque } T \in C^2$$

$$= \iint_D \left(\frac{\partial P\circ T}{\partial u}\frac{\partial T_1}{\partial v} - \frac{\partial P\circ T}{\partial v}\frac{\partial T_1}{\partial u} \right) du\,dv$$

$$= \iint_D \left(\cancel{\frac{\partial P}{\partial x}\frac{\partial T_1}{\partial u}\frac{\partial T_1}{\partial v}} + \frac{\partial P}{\partial y}\frac{\partial T_2}{\partial u}\frac{\partial T_1}{\partial v} + \frac{\partial P}{\partial z}\frac{\partial T_3}{\partial u}\frac{\partial T_1}{\partial v} - \cancel{\frac{\partial P}{\partial x}\frac{\partial T_1}{\partial v}\frac{\partial T_1}{\partial u}} - \frac{\partial P}{\partial y}\frac{\partial T_2}{\partial v}\frac{\partial T_1}{\partial u} - \frac{\partial P}{\partial z}\frac{\partial T_3}{\partial v}\frac{\partial T_1}{\partial u} \right) du\,dv$$

$$= \iint_D \left(\frac{\partial P}{\partial z}\left(\frac{\partial T_3}{\partial u}\frac{\partial T_1}{\partial v} - \frac{\partial T_1}{\partial u}\frac{\partial T_3}{\partial v} \right) - \frac{\partial P}{\partial y}\left(\frac{\partial T_1}{\partial u}\frac{\partial T_2}{\partial v} - \frac{\partial T_2}{\partial u}\frac{\partial T_1}{\partial v} \right) \right) du\,dv$$

$$\Rightarrow \iint_S \left(0, \frac{\partial P}{\partial z}, -\frac{\partial P}{\partial y}\right) ds = \int_{\partial S^+} (P, 0, 0)\,ds$$

Análogamente se puede demostrar que: $\iint_S \left(-\frac{\partial Q}{\partial z}, 0, \frac{\partial Q}{\partial x}\right) ds = \int_{\partial S^+} (0, Q, 0)\,ds$

$$\iint_S \left(\frac{\partial R}{\partial y}, -\frac{\partial R}{\partial x}, 0\right) ds = \int_{\partial S^+} (0, 0, R)\,ds$$

$$\Rightarrow \iint_S \nabla \times F = \int_{\partial S^+} F\,ds \qquad \text{Q.E.D.}$$

TEOREMA DE LA DIVERGENCIA

Sea $\Omega \subseteq \mathbb{R}^3$ una región de Tipo $\underline{\text{IV}}$, $\partial\Omega$ una superficie cerrada con normal exterior y F un campo vectorial diferenciable

$$\Rightarrow \iint_{\partial\Omega} F\,ds = \iiint_{\Omega} \nabla\cdot F\,dxdydz.$$

Tomo: $F=(P,Q,R) \Rightarrow \nabla\cdot F = \frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}+\frac{\partial R}{\partial z}$

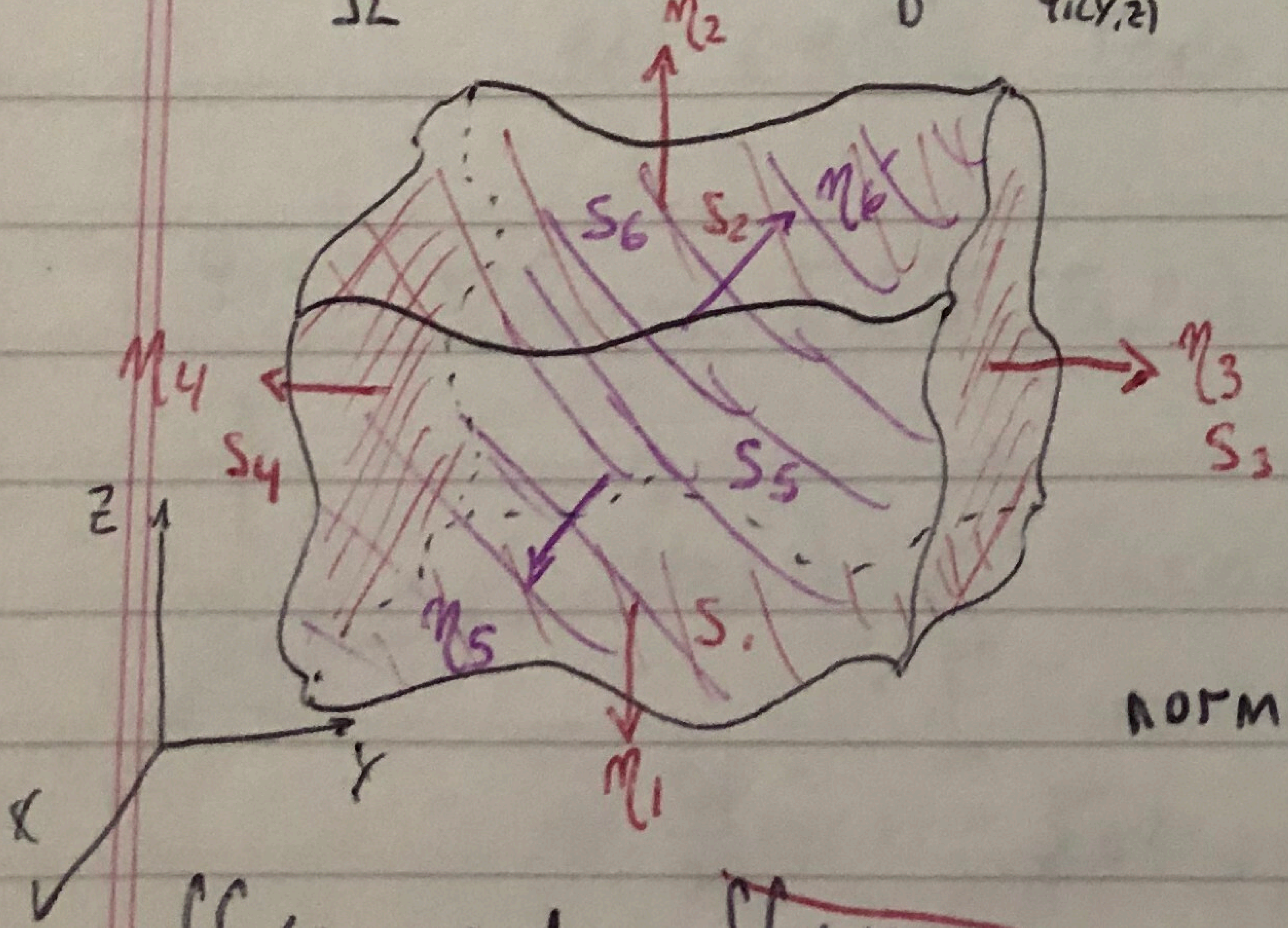
qvq.: $\iint_{\partial\Omega}(P,0,0)\,ds = \iiint_{\Omega}\frac{\partial P}{\partial x}\,dxdydz$, $\iint_{\partial\Omega}(0,Q,0)\,ds = \iiint_{\Omega}\frac{\partial Q}{\partial y}\,dxdydz$,

$$\iint_{\partial\Omega}(0,0,R)\,ds = \iiint_{\Omega}\frac{\partial R}{\partial z}\,dxdydz.$$

Como Ω es una Región de Tipo $\underline{\text{IV}}$ puedo tomarla como:

$$\Omega = \{(x,y,z)\in\mathbb{R}^3 \,/\, (y,z)\in D,\ x\in[\varphi_1(y,z),\varphi_2(y,z)]\}$$

$$\Rightarrow \iiint_{\Omega}\frac{\partial P}{\partial x}\,dxdydz = \iint_{D}\left(\int_{\varphi_1(y,z)}^{\varphi_2(y,z)}\frac{\partial P}{\partial x}\,dx\right)dydz = \underline{\iint_{D}\left(P(\varphi_2(y,z),y,z)-P(\varphi_1(y,z),y,z)\right)dydz}$$



$\partial\Omega = S_1+S_2+S_3+S_4+S_5$

S_5 y S_6 tienen normales paralelas al eje x mientras que las otras tienen normales perpendiculares al eje x.

$$\iint_{\partial\Omega}(P,0,0)\,ds = \iint_{S_1}\cancel{\langle(P,0,0),(0,0,-1)\rangle}\,ds + \iint_{S_2}\cancel{\langle(P,0,0),(0,0,1)\rangle}\,ds$$

$$+\iint_{S_3}\cancel{\langle(P,0,0),(0,1,0)\rangle}\,ds+\iint_{S_4}\cancel{\langle(P,0,0),(0,-1,0)\rangle}\,ds+\iint_{S_5}\langle(P,0,0),(1,0,0)\rangle\,ds+\iint_{S_6}\langle(P,0,0),(-1,0,0)\rangle\,ds$$

Las integrales de las caras perpendiculares se eliminan ya que el producto interno con la normal es nulo.

$$\Rightarrow \iint_{\partial\Omega} (P,0,0)\,ds = \iint_{S_5} \langle (P,0,0),(1,0,0)\rangle ds + \iint_{S_6} \langle (P,0,0),(-1,0,0)\rangle ds$$

$$= \iint_{D} P(\varphi_2(x,z),y,z)\,dy\,dz - \iint_{D} P(\varphi_1(y,z),y,z)\,dy\,dz$$

$$\Rightarrow \iiint_{\Omega} \frac{\partial P}{\partial x}\,dx\,dy\,dz = \iint_{\partial\Omega} (P,0,0)\,ds$$

Análogamente se puede repetir la demostración para:

$$\iiint_{\Omega} \frac{\partial Q}{\partial y}\,dx\,dy\,dz = \iint_{\partial\Omega} (0,Q,0)\,ds \quad y \quad \iiint_{\Omega} \frac{\partial R}{\partial z}\,dx\,dy\,dz = \iint_{\partial\Omega} (0,0,R)\,ds$$

Tomando $\Omega = \{(x,y,z)\in\mathbb{R}^3 \,/\, (x,z)\in\tilde{D},\ y\in[\psi_1(x,z),\psi_2(x,z)]\}$ y

$\Omega = \{(x,y,z)\in\mathbb{R}^3 \,/\, (x,y)\in\hat{D},\ z\in[\delta_1(x,y),\delta_2(x,y)]\}$ respectivamente.

$$\Rightarrow \iint_{\partial\Omega} F\,ds = \iiint_{\Omega} \nabla\cdot F\,dx\,dy\,dz \qquad \text{Q.E.D.}$$

CAMPOS CONSERVATIVOS

Sea F un campo vectorial, $F \in C^1$ salvo quizás en finitos puntos entonces es conservativo si:

① $\oint F\,ds = 0$.

② $\int_{C_1} F\,ds = \int_{C_2} F\,ds$ para C_1 y C_2 curvas que empiezan y terminan en los mismos puntos.

③ $F = \nabla f$, $f: \mathbb{R}^3 \to \mathbb{R}$.

④ $\nabla \times F = \vec{0}$.

Si F cumple alguna de estas condiciones entonces cumple todas.

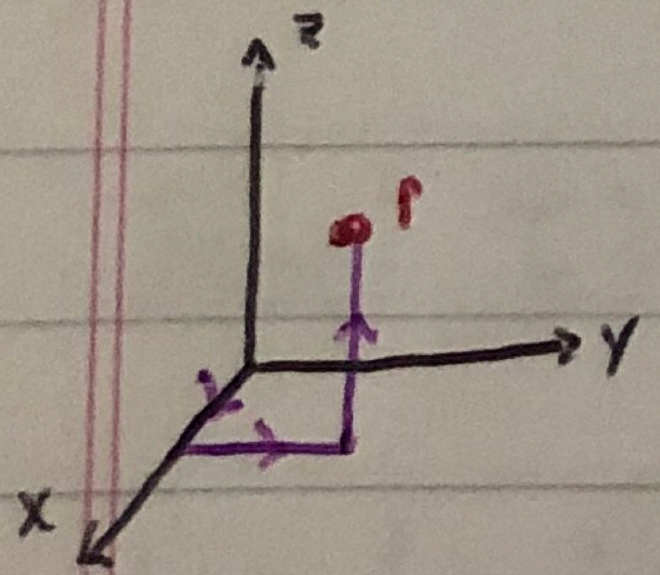
①⇒②: $\int_{C_1^+} F\,ds = \int_{C_2^+} F\,ds$, Tomo: $\tilde{C} = C_1^+ \cup C_2^-$, $\tilde{C}$ es una curva cerrada.

$$\Rightarrow \int_{\tilde{C}} F\,ds = \int_{C_1^+} F\,ds + \int_{C_2^-} F\,ds = \int_{C_1^+} F\,ds - \int_{C_2^+} F\,ds = 0.$$

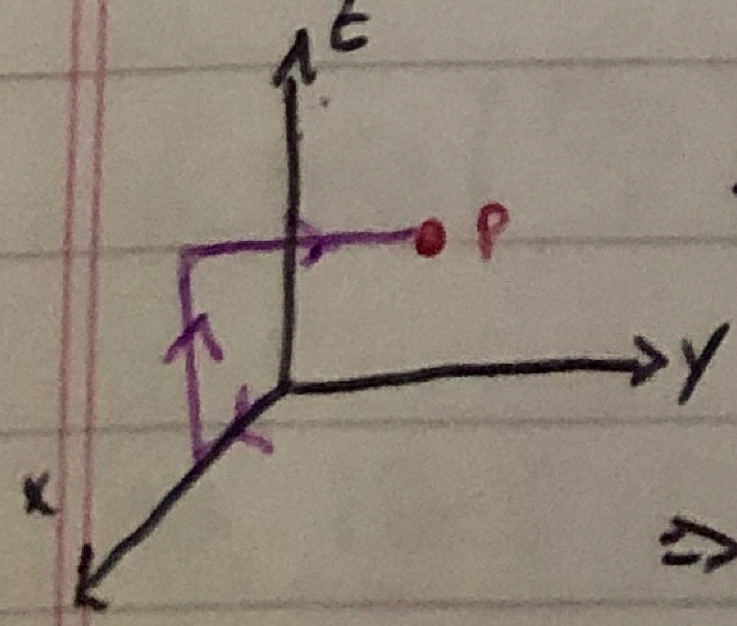
Si C_1 y C_2 fueran a intersecarse en finitos puntos entonces tomo las curvas entre los puntos para demostrarlo así.

②⇒③: Tomemos una curva C que empieza en el origen y termina en $p \in \mathbb{R}^3$. Tomo: $f(x,y,z) = \int_{C^+} F\,ds$

Qvq.: $\nabla f(x,y,z) = F = (F_1, F_2, F_3)$

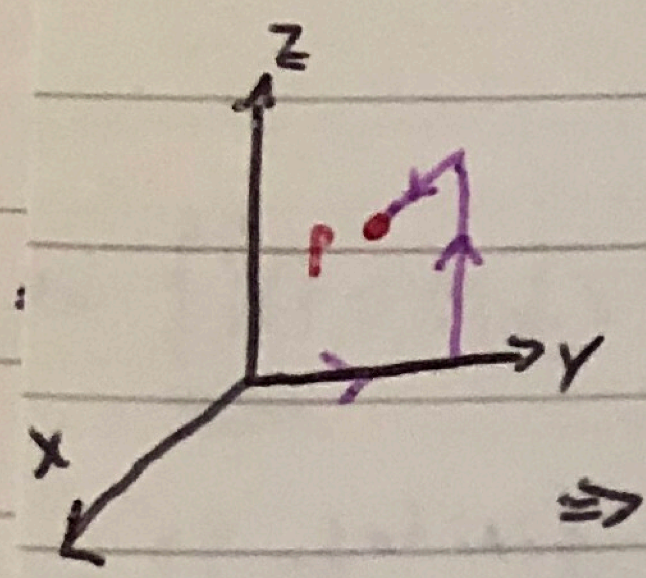


$$f(x,y,z) = \int_0^x \langle F(t,0,0), (1,0,0)\rangle dt + \int_0^y \langle F(x,t,0), (0,1,0)\rangle dt + \int_0^z \langle F(x,y,t), (0,0,1)\rangle dt$$
$$= \int_0^x F_1(t,0,0)\,dt + \int_0^y F_2(x,t,0)\,dt + \int_0^z F_3(x,y,t)\,dt$$
$$\Rightarrow \frac{\partial f}{\partial z}(x,y,z) = F_3(x,y,z)$$ Por el Teorema Fundamental del Cálculo



$$f(x,y,z) = \int_0^x \langle F(t,0,0), (1,0,0)\rangle dt + \int_0^y \langle F(x,t,z), (0,1,0)\rangle dt + \int_0^z \langle F(x,0,t), (0,0,1)\rangle dt$$
$$= \int_0^x F_1(t)\,dt + \int_0^y F_2(x,t,z)\,dt + \int_0^z F_3(x,0,t)\,dt$$
$$\Rightarrow \frac{\partial f}{\partial y}(x,y,z) = F_2(x,y,z)$$ Por el TFC.

Rivadavia



$$f(x,y,z) = \int_0^x \langle F(t,y,z),(1,0,0)\rangle dt + \int_0^y \langle F(0,t,0),(0,1,0)\rangle dt + \int_0^z \langle F(0,y,t),(0,0,1)\rangle dt$$
$$= \int_0^x F_1(t,y,z)\,dt + \int_0^y F_2(0,t,0)\,dt + \int_0^z F_3(0,y,t)\,dt$$
$$\Rightarrow \frac{\partial f}{\partial x}(x,y,z) = F_1(x,y,z)$$ Por el TFC.

$$\Rightarrow \nabla f = F$$

③⇒④: $F = \nabla f \Rightarrow \nabla \times F = \nabla \times (\nabla f) = \left(\frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial z \partial y}, \frac{\partial^2 f}{\partial z \partial x} - \frac{\partial^2 f}{\partial x \partial z}, \frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y \partial x}\right)$

Como $F \in C^1 \Rightarrow$ las derivadas cruzadas de f son iguales
$\Rightarrow \nabla \times F = \vec{0}$

④⇒①: Tomemos C una curva cerrada simple y orientada que encierre una superficie S donde F sea C^1. Por el Teorema de Stokes:

$$\iint_S \nabla \times F\, dS = \oint F\, ds \ . \quad \nabla \times F = \vec{0} \Rightarrow \oint F\, ds = 0$$

Como demostré ①⇒②⇒③⇒④⇒① las propiedades están conectadas y entonces si una se cumple entonces las cuatro deben cumplirse. Q.E.D.

TEOREMA DE EXISTENCIA

Sea $I \subseteq \mathbb{R}$, $f: I \to \mathbb{R}$ Lipschitz en x, $r \in I$, $\xi \in \mathbb{R}$:
$\Rightarrow \exists \lambda > 0$, $x: [r-\lambda, r+\lambda] \to \mathbb{R}$, $x \in C$ /
$x'(t) = f(t, x(t)) \; \forall t \in [r-\lambda, r+\lambda]$
$x(r) = \xi$

f es Lipschitz $\Rightarrow$ f es continua y $\exists L > 0 / |f(t,x) - f(t,y)| \leq L|x-y|$
$\forall x, y \in \mathbb{R}$, $t \in I$.

Tomo: $\lambda_0 = \frac{1}{2L}$, $\lambda = \begin{cases} \lambda_0 & , [r-\lambda_0, r+\lambda_0] \subseteq I \\ m \in (0, \lambda_0) / [r-m, r+m] \subseteq I & , [r-\lambda_0, r+\lambda_0] \nsubseteq I \end{cases}$

Defino: $X_{k+1}(t) = \xi + \int_r^t f(s, x_k(s))\,ds$, $\{X_k\}_{k \in \mathbb{N}}$ una sucesión de funciones continuas, $X_k: [r-\lambda, r+\lambda] \to \mathbb{R}$

$T(y(t)) = \xi + \int_r^t f(s, y(s))\,ds$ un operador.

$\Rightarrow X_{k+1}(t) = T(X_k(t))$, qvq.: X_k es Uniformemente de Cauchy.

Sea $M_k = \text{máx}(|X_{k+1}(t) - X_k(t)|) \Rightarrow M_k = \text{máx}\left(\left|\int_r^t (f(s, x_k(s)) - f(s, x_{k-1}(s)))\,ds\right|\right)$

Tomo: $t \geq r \Rightarrow M_k \leq \text{máx}\left(\int_r^t |f(s, x_k(s)) - f(s, x_{k-1}(s))|\,ds\right)$

f es Lipschitz $\Rightarrow M_k \leq \text{máx}\left(\int_r^t L|X_k(s) - X_{k-1}(s)|\,ds\right)$
$\leq \text{máx}(L M_{k-1}(t-r)) \leq \text{máx}(L M_{k-1} \lambda)$
$\leq \text{máx}\left(\frac{M_{k-1}}{2}\right) \Rightarrow M_k \leq \frac{1}{2} M_{k-1} \; \forall k \in \mathbb{N}$

$\Rightarrow M_k \leq M_0 \left(\frac{1}{2}\right)^k$ A medida que k crece los máximos decrecen.

Rivadavia

$\underline{\text{qvq}}$: $|X_n(t) - X_m(t)| \leq \varepsilon \ \forall t \in I$, $n \geq m$. Para ver si es Uniformemente de Cauchy.

$$\Rightarrow |X_n(t) - X_m(t)| = |X_n(t) - X_{n-1}(t) + X_{n-1}(t) - X_{n-2}(t) + X_{n-2}(t) - \dots + X_{m+1}(t) - X_m(t)|$$
$$\leq |X_n(t) - X_{n-1}(t)| + |X_{n-1}(t) - X_{n-2}(t)| + \dots + |X_{m+1}(t) - X_m(t)| \leq M_{n-1} + M_{n-2} + \dots + M_m$$
$$\leq M_0\left(\frac{1}{2^{n-1}} + \frac{1}{2^{n-2}} + \dots + \frac{1}{2^m}\right) = M_0\left(\sum_{j=0}^{n-1}\left(\tfrac{1}{2}\right)^j - \sum_{j=0}^{m-1}\left(\tfrac{1}{2}\right)^j\right) = M_0\left(\frac{\frac{1}{2^m} - \frac{1}{2^n}}{\frac{1}{2}}\right)$$
$$= M_0\left(\frac{1}{2^{m-1}} - \frac{1}{2^{n-1}}\right) \leq \frac{M_0}{2^{m-1}} = \underline{\frac{M_0}{2^{n_0-1}} < \varepsilon} \Rightarrow X_k \text{ es Uniformemente de Cauchy.}$$

$\underline{\text{Tomo:}}$ $X(t) = \lim_{k\to\infty} X_k(t) = \lim_{k\to\infty} \left(\xi + \int_r^t f(s, X_k(s))\,ds\right) = \underline{\xi + \int_r^t f(s, X(s))\,ds}$

porque f es continua

$$\Rightarrow X(t) = \xi + \int_r^t f(s, X(s))\,ds \quad , \begin{cases} X(r) = \xi \\ X'(t) = f(t, X(t)) \end{cases}$$

Q.E.D.

Por Teorema Fundamental del cálculo

Análogamente se repite la demostración para $t \leq r$.

LEMA DE GRONWAL

Sea $g: I \subseteq \mathbb{R} \to \mathbb{R}$, $a \in I$, $g \geq 0$ y $g(t) \leq A + B\left|\int_a^t g(s)\,ds\right|$ $\forall t \in I$, $A, B > 0 \Rightarrow g(t) \leq Ae^{B|t-a|}$ $\forall t \in I$.

Tomo: $t \geq a$, $G(t) = \int_a^t g(s)\,ds \Rightarrow G'(t) = g(t)$

$G'(t) \leq A + B\int_a^t g(s)\,ds$, Tomo: $H(t) = G(t)\,e^{-B(t-a)}$

$$\Rightarrow H'(t) = G'(t)\,e^{-B(t-a)} - G(t)\,B\,e^{-B(t-a)} = e^{-B(t-a)}\left(G'(t) - B\,G(t)\right)$$

$$\leq e^{-B(t-a)}\left(A + B\int_a^t g(s)\,ds - B\int_a^t g(s)\,ds\right) = Ae^{-B(t-a)}$$

$$\Rightarrow \int_a^t H'(s)\,ds \leq A\int_a^t e^{-B(s-a)}\,ds = \frac{A}{B}\left(1 - e^{-B(t-a)}\right)$$

$$\Rightarrow G(t)\,e^{-B(t-a)} \leq \frac{A}{B}\left(1 - e^{-B(t-a)}\right) \Rightarrow G(t) \leq \frac{A}{B}\left(e^{B(t-a)} - 1\right)$$

$$\Rightarrow \int_a^t g(s)\,ds \leq \frac{A}{B}\left(e^{B(t-a)} - 1\right) \Rightarrow g(t) \leq A + B\left(\frac{A}{B}\left(e^{B(t-a)} - 1\right)\right)$$

$$\Rightarrow \underline{g(t) \leq Ae^{B(t-a)}}$$

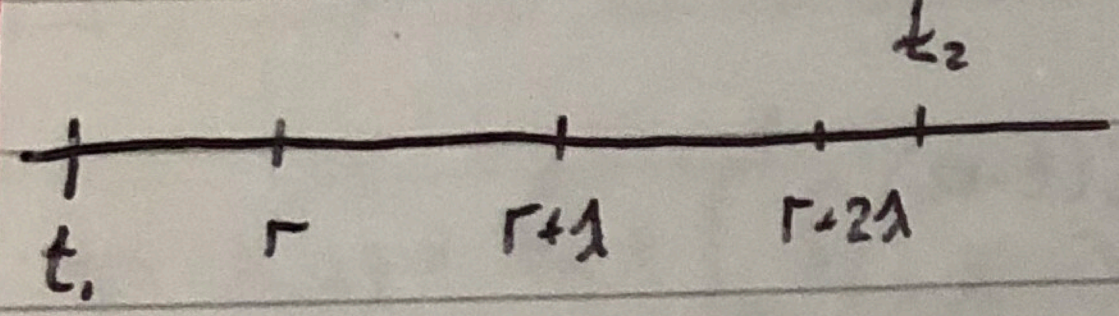
Análogamente se repite la demostración para $t \leq a$. Q.E.D.

Propiedad de Globalidad de la Solución Maximal

Sea $x'(t)=f(t,x(t))$, $x(r)=\xi$, $x: I\subseteq\mathbb{R}\to\mathbb{R}$, $x\in C^1$, f Lipschitz en un intervalo $[t_1, t_2]\subseteq I$ $\Rightarrow$ La solución maximal del sistema es global.

Tomo: $r\in[t_1, t_2]$, $t_1 < t_2$

Como f es Lipschitz $\Rightarrow$ $x_1'(t)=f(t,x_1(t))$ $\forall t\in[r, r+\lambda]$, $\lambda\leq\frac{1}{2L}$.
$x_1(r)=\xi$



Busquemos un $x'=f(t,x)$ / $x(r+\lambda)=x_1(r+\lambda)$

$$\Rightarrow \begin{cases} x'(t)=f(t,x(t)) \\ x(r+\lambda)=x_1(r+\lambda) \end{cases}$$

Existe solución $x_2: [r+\lambda, r+2\lambda]\to\mathbb{R}$

Tomo: $$\tilde{x}_1(t)=\begin{cases} x_1(t), & t\in[r, r+\lambda] \\ x_2(t), & t\in[r+\lambda, r+2\lambda] \end{cases}$$, $\tilde{x}_1(t)$ es solución.

Si $r+2\lambda < t_2$ $\Rightarrow$ Repito el proceso.

Sabemos que el proceso va a terminar ya que $\exists k\in\mathbb{N} / r+k\lambda\geq t_2$ así que logramos extender la solución hasta t_2. Análogamente se puede extender la solución a t_1 y entonces la solución es global.

Q.E.D.

TEOREMA DE ESPACIO VECTORIAL DE SOLUCIONES

Sea $I \subseteq \mathbb{R}$ cerrado, $A: I \to \mathbb{R}^{n \times n}$ con coeficientes continuos
$\Rightarrow$ Las soluciones del sistema $X'(t) = A(t)X(t)$, $X: I \to \mathbb{R}^n$ existen en un espacio vectorial de dimensión n.

Tomo: $S = \{X: I \to \mathbb{R}^n / X$ es solución de $X' = AX\}$

qvq.: S es un espacio vectorial y que $\dim(S) = n$

$0 \in S$? ✓ $0 = A \cdot 0$, Tomo: $X_1, X_2 \in S$

$X_1 + X_2 \in S$? ✓ $(X_1 + X_2)' = X_1' + X_2' = AX_1 + AX_2 = A(X_1 + X_2)$

Tomo: $\lambda \in \mathbb{R}$, $\lambda X_1 \in S$? ✓ $(\lambda X_1)' = \lambda X_1' = \lambda A X_1 = A(\lambda X_1)$

$\Rightarrow$ S es un espacio vectorial.

$\forall i \in [1, n] \subseteq \mathbb{N}$ Tomo $X_i: I \to \mathbb{R}^n$ la única solución de
$\begin{cases} X'(t) = A(t)X(t) \\ X(r) = e_i \end{cases}$, $r \in I$, e_i el vector canónico i.

$\Rightarrow B_S = \{X_i\}$, qvq.: El conjunto $\{X_i\}$ es L.I.

$\alpha_1 X_1 + \alpha_2 X_2 + \dots + \alpha_n X_n = 0 \ \forall t \in I$, $\alpha_1, \alpha_2, \dots, \alpha_n \in \mathbb{R}$

$t = r \Rightarrow \alpha_1 X_1 + \alpha_2 X_2 + \dots + \alpha_n X_n = \alpha_1 e_1 + \alpha_2 e_2 + \dots + \alpha_n e_n = 0 \Rightarrow \alpha_1, \alpha_2, \dots, \alpha_n = 0$

Sea $X \in S$ qvq.: $\exists c_i / X = \sum_{i=1}^{n} c_i X_i$

Tomo: $X(r) = \xi \Rightarrow X(r) = \xi_1 e_1 + \xi_2 e_2 + \dots + \xi_n e_n$

$= \xi_1 X_1(r) + \xi_2 X_2(r) + \dots + \xi_n X_n(r)$

Tomo: $Y(t) = \sum_{i=1}^{n} \xi_i X_i(t)$, qvq.: $Y(t) = X(t) \ \forall t \in I$

$y \in S$? ✓ Porque es una Combinación Lineal de soluciones.

$Y(r) = \xi = X(r)$

$\Rightarrow X(t) = Y(t)$ porque existe una única solución.

$\Rightarrow \{X_i\}$ es un conjunto L.I. ya que cualquier solución se puede escribir como una combinación lineal de $\{X_i\}$

$\Rightarrow B = \{X_i\}$ es base de soluciones de S con n elementos

$\Rightarrow S$ es un espacio vectorial de dimensión n. Q.E.D.

TEOREMA DE UNICIDAD

Sea $f: I \subseteq \mathbb{R} \to \mathbb{R}$ lipschitz, $r \in I$, $\xi_1, \xi_2 \in \mathbb{R}$, $\eta > r \in I$

$x_i : [r, \eta] \to \mathbb{R}$ C^1 / $x_i'(t) = f(t, x_i(t))$ $\forall t \in [r, \eta]$, $i \in [1, 2] \subseteq \mathbb{N}$

$$x_i(r) = \xi_i$$

$\Rightarrow \exists\, C(\eta) > 0 \,/\, |x_1(t) - x_2(t)| \leq C(\eta)|\xi_1 - \xi_2|.$

$x_i'(t) = f(t, x_i(t))$ $\Rightarrow \int_r^t x_i'(s)\,ds = \int_r^t f(s, x_i(s))\,ds = x_i(t) - x_i(r)$

$x_i(r) = \xi_i$

$$\Rightarrow x_i(t) = x_i(r) + \int_r^t f(s, x_i(s))\,ds = \xi_i + \int_r^t f(s, x_i(s))\,ds$$

$\Rightarrow x_1(t) = \xi_1 + \int_r^t f(s, x_1(s))\,ds$, **Tomo:** $f(s, x_i(s)) = f_i$

$x_2(t) = \xi_2 + \int_r^t f(s, x_2(s))\,ds$

$$\Rightarrow |x_1(t) - x_2(t)| = |\xi_1 + \int_r^t f_1 - \xi_2 - \int_r^t f_2| = |(\xi_1 - \xi_2) + \int_r^t (f_1 - f_2)\,ds|$$

Como f es lipschitz $\Rightarrow |f_1 - f_2| \leq L|x_1(t) - x_2(t)|$

$$\Rightarrow |x_1(t) - x_2(t)| \leq |\xi_1 - \xi_2| + \int_r^t |f_1 - f_2|\,ds$$

$$|x_1(t) - x_2(t)| \leq |\xi_1 - \xi_2| + L\int_r^t |x_1(t) - x_2(t)|\,ds$$

Por Gronwal: $|x_1(t) - x_2(t)| \leq |\xi_1 - \xi_2|\, e^{L|t-r|}$

$$\Rightarrow |x_1(t) - x_2(t)| \leq \underbrace{e^{L|\eta - r|}}_{C(\eta)} |\xi_1 - \xi_2|$$

Q.E.D.

LEMA DE SISTEMA DE ECUACIONES LIPSCHITZ

Sea el sistema $F(t,x) = A(t)X(t) + b(t)$ con $A: I \subseteq \mathbb{R} \to \mathbb{R}^{n\times n}$ continua, $b: I \to \mathbb{R}^n$ continua y $X: I \to \mathbb{R}^n$ C^1, si I es cerrado y acotado $\Rightarrow F$ es lipschitz en X.

qvq.: $\|F(t,X(t)) - F(t,Y(t))\| \leq L\|X(t) - Y(t)\|$

$$\|F(t,x) - F(t,y)\|^2 = \|A(x-y)\|^2 = \sum_{j=1}^{n} (A(x-y))_j^2$$

$$|(A(x-y))_j| = \left|\sum_{i=1}^{n} (a_{ji}(x_i - y_i))\right| \leq \left(\sum_{i=1}^{n} a_{ji}^2\right)^{\frac{1}{2}} \left(\sum_{i=1}^{n} (x_i - y_i)^2\right)^{\frac{1}{2}}$$

$$\Rightarrow \sum_{j=1}^{n} (A(x-y))_j^2 \leq \|x-y\|^2 \sum_{i,j=1}^{n} a_{ji}^2$$

Como A es continua e I es compacto $\Rightarrow |a_{ji}| \leq K \ \forall t$

$$\Rightarrow \|F(t,x) - F(t,y)\|^2 \leq K^2 n^2 \|x-y\|^2$$

$$\Rightarrow \|F(t,x) - F(t,y)\| \leq \underbrace{Kn}_{L} \|x-y\|$$

$\|F(t,X(t)) - F(t,Y(t))\| \leq L\|X(t) - Y(t)\|$, F es lipschitz en X.

Q.E.D.

Propiedad de Soluciones Linealmente Independientes

Sean $\{X_i\}$, $i \in [1,n] \subseteq \mathbb{N}$ un conjunto de soluciones L.I. del sistema $X'_{(t)} = A_{(t)} X_{(t)}$, $t \in I \subseteq \mathbb{R}$, $X, X_i : I \to \mathbb{R}^n$, $A : I \to \mathbb{R}^{n\times n}$ continuo, $r \in I \iff X_i(r)$ son L.I.

$\Rightarrow$) Tomo: $X_{(t)} = \sum_{i=1}^{n} \alpha_i X_{i(t)}$, $\alpha_i \in \mathbb{R} / \sum_{i=1}^{n} \alpha_i X_{i(r)} = 0$

$\Rightarrow X_{(r)} = 0$, qvq: $\alpha_i = 0 \; \forall i$

X es solución única así que por unicidad $X_{(t)} = 0 \; \forall t \in I$.
porque 0 es solución

$\Rightarrow \sum_{i=1}^{n} \alpha_i X_{i(t)} = 0 \; \forall t \in I$. Como X_i son L.I. $\Rightarrow \alpha_i = 0 \; \forall i$

$\Rightarrow X_i(r)$ son L.I.

$\Leftarrow$) Tomo: $\alpha_i \in \mathbb{R} / \sum_{i=1}^{n} \alpha_i X_{i(t)} = 0 \; \forall t \in I$

$t = r$: $\sum_{i=1}^{n} \alpha_i X_{i(r)} = 0$, Como $X_{i(r)}$ son L.I. $\Rightarrow \alpha_i = 0 \; \forall i$

$\Rightarrow X_{i(t)}$ son L.I. $\forall t \in I$. Q.E.D.

TEOREMA DE SOLUCIÓN HOMOGÉNEA Y PARTICULAR

Sea el sistema $X'(t) = A(t)X(t) + b(t)$, $A: I \subseteq \mathbb{R} \to \mathbb{R}^{n \times n}$ continua, $b: I \to \mathbb{R}^n$ continua, $X: I \to \mathbb{R}^n$ C^1, sea $X_p(t)$ una solución del sistema y $X_h(t)$ la solución del sistema homogéneo asociado

$\Leftrightarrow$ $S = \{X: I \to \mathbb{R}^n / X(t) = X_p(t) + X_h(t)\}$ es el conjunto de soluciones del sistema.

$\Rightarrow)$ Tomo: $Y(t) = X(t) - X_p(t)$, $X(t)$ solución del sistema

$$\Rightarrow Y'(t) = X'(t) - X_p'(t) = A(t)X(t) - A(t)X_p(t) = A(t)Y(t)$$

$Y(t)$ es solución del sistema homogéneo asociado.

$$\Rightarrow Y(t) = X_h(t) = X(t) - X_p(t) \Rightarrow X(t) = X_p(t) + X_h(t)$$

$\Leftarrow)$ $X(t) = X_h(t) + X_p(t) \Rightarrow X'(t) = X_h'(t) + X_p'(t) = A(t)X_h(t) + A(t)X_p(t) + b(t)$

$= A(t)X(t) + b(t) \Rightarrow X(t)$ es solución del sistema.

Q.E.D.

TEOREMA DE LA SOLUCIÓN PARTICULAR

Sea $\{X_i\}$, $i \in [1,n] \subseteq \mathbb{N}$ una base de soluciones del sistema homogéneo asociado a $X'_{(t)} = A_{(t)} X_{(t)} + b_{(t)}$, $A: I \subseteq \mathbb{R} \to \mathbb{R}^{n \times n}$ continua, $b: I \to \mathbb{R}^n$ continua, $X: I \to \mathbb{R}^n$ C^1 $\Rightarrow \exists\, C_i: I \to \mathbb{R}^n$ C^1 / $X_{p(t)} = \sum_{i=1}^{n} C_{i(t)} X_{i(t)}$ sea una solución del sistema. Cumplen que $Q_{(t)} (C_{i(t)})' = b_{(t)}$ donde $Q_{(t)}$ es la matriz fundamental.

Tomo: $X_{(t)} = \sum_{i=1}^{n} C_{i(t)} X_{i(t)} \Rightarrow X_{(t)} = Q_{(t)} C_{(t)}$

ya que $Q_{(t)} = \begin{pmatrix} | \\ X_i \\ | \end{pmatrix}$ y $C_{(t)} = \begin{pmatrix} | \\ C_i \\ | \end{pmatrix}$, $Q_{(t)} \in \mathbb{R}^{n \times n}$, $C_{(t)} \in \mathbb{R}^n$.

$\Rightarrow X'_{(t)} = Q'_{(t)} C_{(t)} + Q_{(t)} C'_{(t)}$ *, $Q'_{(t)} = A_{(t)} Q_{(t)}$

$\Rightarrow X'_{(t)} = A_{(t)} Q_{(t)} C_{(t)} + Q_{(t)} C'_{(t)} = A_{(t)} X_{(t)} + \underbrace{Q_{(t)} C'_{(t)}}_{b_{(t)}}$

qvq.: $Q_{(t)} C'_{(t)} = b_{(t)} \Rightarrow X_{(t)}$ es solución.

$C'_{(t)} = Q^{-1}_{(t)} b_{(t)}$ necesito que exista $Q^{-1}_{(t)}$.

Como $Q_{(t)}$ está formado por vectores L.I. $\Rightarrow \det(Q_{(t)}) \neq 0 \Rightarrow \exists\, Q^{-1}_{(t)}$.

Como $X_{i(t)}$ son continuas $\Rightarrow Q_{(t)}$ es continua $\Rightarrow Q^{-1}_{(t)}$ es continua.

$\Rightarrow C'_{(t)} = Q^{-1}_{(t)} b_{(t)}$ es continua $\Rightarrow C_{(t)}$ es C^1. Q.E.D.

LEMA DEL PUNTO DE EQUILIBRIO

Sea $X_{(t)}$ una solución de $X'_{(t)} = F(X_{(t)})$, $X: I \subseteq \mathbb{R} \to \mathbb{R}^n$:

① Si $t_0 \in I / X(t_0) = X_0$, $X_{(t)} \neq X_0 \ \forall t \in I \setminus \{t\} \Rightarrow F(X_0) \neq 0$.

② Si $\lim_{t \to \pm\infty} X_{(t)} = X_0 \Rightarrow F(X_0) = 0$.

① Supongamos que $F(X_0) = 0$; $\tilde{X}_{(t)} = X_0 \ \forall t$ es solución

Por unicidad $\Rightarrow \tilde{X}_{(t)} = X_{(t)} = cte.$ ABSURDO!

$\Rightarrow \underline{F(X_0) \neq 0}$.

② $\underline{n=2}$: $X_{(t)} = \begin{pmatrix} X_1(t) \\ X_2(t) \end{pmatrix}$, $F(X_{(t)}) = F(X_1(t), X_2(t)) = (f_1(X_1(t), X_2(t)), f_2(X_1(t), X_2(t)))$.

$X(t+1) - X(t) = \begin{pmatrix} X_1(t+1) - X_1(t) \\ X_2(t+1) - X_2(t) \end{pmatrix} = \begin{pmatrix} X_1'(\lambda_1) \\ X_2'(\lambda_2) \end{pmatrix}$, $\lambda_1, \lambda_2 \in [t, t+1]$ Por Teorema de Valor Medio.

$\Rightarrow X(t+1) - X(t) = \begin{pmatrix} f_1(X_1(\lambda_1), X_2(\lambda_1)) \\ f_2(X_1(\lambda_2), X_2(\lambda_2)) \end{pmatrix}$

$$\lim_{t \to \pm\infty} (X(t+1) - X(t)) = \lim_{t \to \pm\infty} \begin{pmatrix} f_1(X_1(\lambda_1), X_2(\lambda_1)) \\ f_2(X_1(\lambda_2), X_2(\lambda_2)) \end{pmatrix} = F(X_0)$$

$$\lim_{t \to \pm\infty} (X(t+1) - X(t)) = \lim_{t \to \pm\infty} X(t+1) - \lim_{t \to \pm\infty} X(t) = X_0 - X_0 = 0$$

$\Rightarrow \underline{F(X_0) = 0}$.

Q.E.D.

Propiedad del Wronskiano

Sea $\{X_i\}$, $i\in[1,n]\subseteq\mathbb{N}$ una base de soluciones del sistema homogéneo asociado a $X^{(n)}_{(t)} + \sum_{i=0}^{n-1} a_{i(t)}X^{(i)}_{(t)} = f_{(t)}$, $X: I\subseteq\mathbb{R}\to\mathbb{R}$, $X\in C^n$, $a_i: I\to\mathbb{R}$ continuas, $f: I\to\mathbb{R}$ continua

$\Rightarrow \exists\, C_i: I\to\mathbb{R}$, $C_i\in C^1$ / $X_{p(t)} = \sum_{i=1}^{n} C_{i(t)}X_{i(t)}$ sea una solución del sistema. Cumplen que $W_{(t)}(C_{i(t)})' = \begin{pmatrix}0\\ f_{(t)}\end{pmatrix}$ donde $W_{(t)}$ es la matriz del Wronskiano.

$n=2$: $X''_{(t)} + a_{1(t)}X'_{(t)} + a_{0(t)}X_{(t)} = f_{(t)}$, $\{X_{1(t)}, X_{2(t)}\}$ Base de soluciones del homogéneo

qvq.: $\exists\, C_{1(t)}, C_{2(t)}$ / $X_{p(t)} = C_{1(t)}X_{1(t)} + C_{2(t)}X_{2(t)}$ sea solución.

$$X_p'(t) = C_{1(t)}'X_{1(t)} + C_{1(t)}X_{1(t)}' + C_{2(t)}'X_{2(t)} + C_{2(t)}X_{2(t)}'$$
$$X_p''(t) = C_{1(t)}''X_{1(t)} + 2C_{1(t)}'X_{1(t)}' + C_{1(t)}X_{1(t)}'' + C_{2(t)}''X_{2(t)} + 2C_{2(t)}'X_{2(t)}' + C_{2(t)}X_{2(t)}''$$

Reemplazo en la ecuación:

$$C_{1(t)}''X_{1(t)} + 2C_{1(t)}'X_{1(t)}' + C_{1(t)}X_{1(t)}'' + C_{2(t)}''X_{2(t)} + 2C_{2(t)}'X_{2(t)}' + C_{2(t)}X_{2(t)}'' + a_{1(t)}C_{1(t)}'X_{1(t)} + a_{1(t)}C_{1(t)}X_{1(t)}' + a_{1(t)}C_{2(t)}'X_{2(t)} + a_{1(t)}C_{2(t)}X_{2(t)}' + a_{0(t)}C_{1(t)}X_{1(t)} + a_{0(t)}C_{2(t)}X_{2(t)} = f_{(t)}$$

$$C_{1(t)}\overbrace{\left(X_{1(t)}'' + a_{1(t)}X_{1(t)}' + a_{0(t)}X_{1(t)}\right)}^{=0} + C_{2(t)}\overbrace{\left(X_{2(t)}'' + a_{1(t)}X_{2(t)}' + a_{0(t)}X_{2(t)}\right)}^{=0} +$$
$$2C_{1(t)}'X_{1(t)}' + a_{1(t)}C_{1(t)}'X_{1(t)} + C_{1(t)}''X_{1(t)} + 2C_{2(t)}'X_{2(t)}' + a_{1(t)}C_{2(t)}'X_{2(t)} + C_{2(t)}''X_{2(t)} = f_{(t)}$$

$$\Rightarrow a_{1(t)}\overbrace{\left(C_{1(t)}'X_{1(t)} + C_{2(t)}'X_{2(t)}\right)}^{X(t)} + \overbrace{\left(C_{1(t)}''X_{1(t)} + C_{1(t)}'X_{1(t)}' + C_{2(t)}''X_{2(t)} + C_{2(t)}'X_{2(t)}'\right)}^{X'(t)}$$
$$+ C_{1(t)}'X_{1(t)}' + C_{2(t)}'X_{2(t)}' = f_{(t)}$$

qvq.:
$$C_{1(t)}'X_{1(t)} + C_{2(t)}'X_{2(t)} = 0$$
$$C_{1(t)}'X_{1(t)}' + C_{2(t)}'X_{2(t)}' = f_{(t)}$$
$$\Rightarrow \overbrace{\begin{pmatrix}X_{1(t)} & X_{2(t)}\\ X_{1(t)}' & X_{2(t)}'\end{pmatrix}}^{W_{(t)}}\begin{pmatrix}C_{1(t)}'\\ C_{2(t)}'\end{pmatrix} = \begin{pmatrix}0\\ f_{(t)}\end{pmatrix}$$

Q.E.D.